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ON MIXED FINITE ELEMENT METHODS FOR FIRST ORDER ELLIPTIC SYSTEMS*

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ABSTRACT

A physically based duality theory for first order elliptic systems is shown to be of central importance in connection with the Galerkin finite element solution of these systems. Using this theory in conjunction with a certain hypothesis concerning approximation spaces, optimal error estimates for Galerkin type approximations are demonstrated. An example of a grid which satisfies the hypothesis is given and numerical examples which illustrate the theory are provided.

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I. Introduction

Mixed finite element methods are finite element approximations based on stationary variational principles as contrasted with those based on extremal principles which yield strict maxima or minima. Remarkable progress has been made in the finite element theory for elliptic boundary value problems, especially for those methods which are based on extremal principles. However, there still exists a gap in the theory as it concerns mixed methods. In particular, error estimates presently available in the literature often predict rates of convergence well below those observed in computations. The purpose of this paper is to develop a sharp theory for mixed finite element methods in the context of approximate solutions to the Poisson equation.

The fact that a particular variational principle is stationary in nature has serious implications for finite element approximations. For example, it is well known that finite element approximations based on the Dirichlet Principle will be, in a suitable sense, unconditionally stable and their convergence depends only on the ability to approximate in the finite element spaces [7]. These are not true for methods based on stationary principles. For instance, for the Galerkin method based on the Kelvin Principle considered in this work, we shall find that to obtain stability and convergence certain conditions must be satisfied which restrict the type of grids that can be used. The theory of the present work contains both necessary and sufficient conditions for the stability and convergence of mixed finite element methods derived from the Kelvin Principle.

We begin by stating the boundary value problem to be considered and some equivalent variational formulations. Let Ω be a bounded region in $\mathbb{R}^n$ whose boundary Γ consists of two parts, Γ_D and Γ_N. Given (*)

(*) The space $H^r(\Omega)$ denotes the Sobolev space of order r, $\|\cdot\|_r$ denotes the norm on this space [1], [4].

$$f_0 \in H^0(\Omega)$$

we seek a real valued function ϕ_0 satisfying

(1) $$\Delta\phi_0 = f_0 \quad \text{in} \quad \Omega$$

(2) $$\phi_0 = 0 \quad \text{on} \quad \Gamma_D$$

(3) $$\nabla\phi_0 \cdot \underline{\nu} = 0 \quad \text{on} \quad \Gamma_N \quad ,$$

where $\underline{\nu}$ is the outer normal to Γ. Alternatively, find ϕ_0 and the $\mathbb{R}^n$ valued function $\underline{u}_0$ satisfying

(4) $$\mathrm{div}(\underline{u}_0) = f_0 \quad \text{in} \quad \Omega$$

(5) $$\nabla\phi_0 - \underline{u}_0 = 0 \quad \text{in} \quad \Omega$$

(6) $$\phi_0 = 0 \quad \text{on} \quad \Gamma_D$$

(7) $$\underline{u}_0 \cdot \underline{\nu} = 0 \quad \text{on} \quad \Gamma_N.$$

The classical Dirichlet Principle uses the spaces

(8) $$S = H^1(\Omega) \quad , \quad S_0 = \{\psi \in S: \psi=0 \text{ on } \Gamma_D\} \; ,$$

and asserts that the solution ϕ_0 of (1)-(3) minimizes

(9) $$\int_\Omega \{\tfrac{1}{2} \nabla\psi\cdot\nabla\psi + f_0\psi\}$$

over $\psi \in S_0$. Observe that if

(10) $$\underline{V}_0 = \nabla S_0$$

this is equivalent to minimizing

$$\int_\Omega \{\tfrac{1}{2}\,\underline{v}\cdot\underline{v} + f_0\psi\}$$

over $(\psi,\underline{v}) \in S_0 \times \underline{V}_0$ subject to the constraint

$$\underline{v} = \nabla\,\psi\ .$$

The Kelvin Principle is in some sense dual to the Dirichlet Principle with div being the dual of ∇. In this setting we let

$$(11)\qquad \underline{V} = \underline{H}^1(\Omega)\quad ,\quad \underline{V}_0 = \{\underline{v} \in \underline{V}\colon\ \underline{v}\cdot\underline{\nu}=0 \text{ on } \Gamma_N\}\quad ,$$

and the Kelvin Principle asserts that $\underline{u}_0$ minimizes

$$\tfrac{1}{2}\int_\Omega \underline{v}\cdot\underline{v}$$

over $\underline{v} \in \underline{V}_0$ subject to

$$\mathrm{div}(\underline{v}) = f_0\ .$$

The scalar ϕ_0 enters into the Kelvin Principle as a Lagrange multiplier, i.e., an equivalent statement of the Kelvin Principle is the following. Let

$$(12)\qquad S_0 = \mathrm{div}(\underline{V}_0)\ ,$$

then find

$$(\phi_0,\underline{u}_0) \in S_0 \times \underline{V}_0$$

satisfying

$$(13)\qquad \int_\Omega \{\underline{u}_0\cdot\underline{v} + \phi_0\mathrm{div}(\underline{v}) + \psi\mathrm{div}(\underline{u}_0)\} = \int_\Omega f_0\psi$$

for all $(\psi,\underline{v}) \in S_0 \times \underline{V}_0$.

In the fluid dynamic context [6] the Dirichlet Principle asserts that among all irrotational fields the one that minimizes the kinetic energy is the incompressible field. Dually, the Kelvin Principle asserts that among all incompressible fields, the field that minimizes the kinetic energy is irrotational.

One uses the Dirichlet Principle in computations as follows. Let for example

$$S^h \subset S$$

denote the space of continuous piecewise linear functions on some triangulation of Ω and let

$$S_0^h = \{\psi^h \in S^h: \psi^h = 0 \text{ on } \Gamma_D\} .$$

Compute the minimum of (9) as ψ ranges over S_0^h instead of all of S_0. If ϕ_h is the point where the minimum is achieved and if $\underline{u}_h = \nabla\phi_h$, then it is well known that

$$\|\phi_0 - \phi_h\|_0 \leq C h^2 \|\phi_0\|_2 \tag{14}$$

$$\|\underline{u}_0 - \underline{u}_h\|_0 \leq C h \|\underline{u}_0\|_1 \tag{15}$$

(see [1], [7]).

The Kelvin principle is in some sense a dual to the Dirichlet principle with the greatest stress being placed on the vector $\underline{u}_0$; i.e., in this method the $\underline{u}_0$ is represented in terms of piecewise linear functions and presumably errors of the form

$$\|\underline{u}_0 - \underline{u}_h\|_0 \leq C h^2 \|\underline{u}_0\|_2 . \tag{16}$$

(17) $$\| \phi_0 - \phi_h \|_0 \leq C h \| \phi_0 \|_1 ,$$

are obtained.

More precisely, we compute $\{\phi_h, \underline{u}_h\}$ by letting

(18) $$\underline{V}^h \subset \underline{V}$$

denote the finite dimensional space of $\mathbb{R}^n$ valued continuous piecewise linear functions, and letting

(19) $$\underline{V}_0^h = \{\underline{v}^h \in \underline{V}^h : \underline{v}^h \cdot \underline{\nu} = 0 \text{ on } \Gamma_N\}$$

and

(20) $$S_0^h = \operatorname{div}(\underline{V}_0^h).$$

The pair

(21) $$\{\phi_h, \underline{u}_h\} \in S_0^h \times \underline{V}_0^h$$

is determined by requiring that (13) hold for all $\{\psi, \underline{v}\} \in S_0^h \times \underline{V}_0^h$ with (21) replacing $\{\phi_0, \underline{u}_0\}$.

Unfortunately, (16)-(17) are in general not true without further conditions on the subspace $\underline{V}_0^h$. In subsequent sections we shall give necessary and sufficient conditions for results of the type (16)-(17) to be valid.

Previous work on this problem [2], [3], [5], [8] is based on the Babuska-Brezzi condition, i.e.,

(22) $$\sup_{\underline{v}^h \in \underline{V}_0^h} \left\{ \frac{\int_\Omega \operatorname{div} \underline{v}^h \psi^h}{\| \underline{v}_h \|_0 + \| \operatorname{div} \underline{v}_h \|_0} \right\} \geq c > 0 \qquad \text{for all } \psi^h \in S_0^h .$$

This type of condition leads to an error estimate of the form

(23) $$\| \underline{u}_0 - \underline{u}_h \|_0 \leqq C\, h \, \| \underline{u}_0 \|_2$$

(24) $$\| \phi_0 - \phi_h \|_0 \leqq C\, h \, \| \underline{u}_0 \|_2$$

for piecewise linear elements. This is clearly unsatisfactory since it implies there is no advantage in using the Kelvin Principle except, perhaps, for the fact that the Dirichlet boundary conditions are natural in this context. Our theory indicates that for a certain class of grids the optimal accuracy (16) is achieved. These grids satisfy the Grid Decomposition Property defined in the next section. The latter is necessary and sufficient for stability and optimal accuracy. Incidentally, there is a dual of this property for the Dirichlet principle, but it reduces to a requirement that the space S^h contains the constant function $\psi^h = 1$. This property is possessed by all known finite element spaces.

These results have been generalized [9] to include other physical situation described by equations related to the Navier-Stokes or Maxwell equations.

II. The Discrete Kelvin Principle

To formulate the discrete approximation we let

(1) $$\underline{V} = \underline{H}^1(\Omega) \quad , \quad \underline{V}_0 = \{\underline{v} \in \underline{V} \colon \underline{v}\cdot\underline{\nu}=0 \text{ on } \Gamma_N\}.$$

The next step is to let

(2) $$\underline{V}^h \subset \underline{V}$$

be a finite dimensional space and

(3) $$\underline{V}_0^h = \{\underline{v}^h \in \underline{V}^h \colon \underline{v}^h\cdot\underline{\nu}=0 \text{ on } \Gamma_N\}.$$

Then letting

(4) $$S_0^h = \text{div}\ \underline{V}_0^h$$

the discrete Kelvin Principle requires us to compute

(5) $$\{\phi_h, \underline{u}_h\} \in S_0^h \times \underline{V}_0^h$$

satisfying

(6) $$\int_\Omega \{\underline{u}_h \cdot \underline{v}^h + \phi_h \text{div}(\underline{v}^h) + \psi^h \text{div}(\underline{u}_h)\} = \int_\Omega \psi^h f_0$$

for all

$$\{\psi^h, \underline{v}^h\} \in S_0^h \times \underline{V}_0^h.$$

Once a basis for $S_0^h \times \underline{V}_0^h$ has been chosen, (6) reduces to a system of N algebraic equtions, where N is the dimension of this space.

We shall assume that $\underline{V}_0^h$ and S_0^h satisfy the following property:

Approximation property. *There is an integer* $k \geq 1$ *and a constant* $0 < C_A < \infty$ *(independent of* h) *such that for each* $\underline{v} \in \underline{V}_0$ *there is a* $\hat{\underline{v}}_h \in \underline{V}_0^h$ *satisfying*

(7) $$\| \underline{v} - \hat{\underline{v}}_h \|_0 \leqq C_A h^k \| \underline{v} \|_k \quad ,$$

and for each $\psi \in S_0$ *there is a* $\hat{\psi}_h \in S_0^h$ *satisfying*

(8) $$\| \psi - \hat{\psi}_h \|_0 \leqq C_A h^{k-1} \| \psi \|_{k-1} \ .$$

In addition we assume that (7)-(8) *hold if* k *is replaced by* k' *for any* $0 < k' \leq k$.

This property is valid for spaces of piecewise polynomial functions of degree k-1. For example, k = 2 with linear elements. The error estimate (7) is standard (see [1], [7]). The space S_0^h in this case is contained in the space of all piecewise constant functions. It may be strictly contained in the latter but there are always enough functions in S_0^h to achieve (8) for any $\psi \in S_0$. This is discussed in Section 5.

We are now prepared to introduce the Grid Decomposition Property. To motivate it let us recall that any $\underline{v} \in \underline{V}_0$ can be decomposed as

(9) $$\underline{v} = \nabla \xi + \underline{z}$$

where

(10) $$\operatorname{div}(\underline{z}) = 0$$

and

(11) $$\int_\Omega \underline{z} \cdot \nabla \xi = 0 .$$

Indeed, we construct ξ by solving

(12) $$\Delta \xi = \operatorname{div}(\underline{v}) \quad \text{in} \quad \Omega$$

(13) $$\xi = 0 \quad \text{on} \quad \Gamma_D$$

(14) $$\nabla \xi \cdot \underline{\nu} = 0 \quad \text{on} \quad \Gamma_N \quad ,$$

and then determine $\underline{z}$ by

(15) $$\underline{z} = \underline{v} - \nabla \xi \quad .$$

Observe that if

(16) $$\underline{w} = \nabla \xi \quad ,$$

then from the theory of partial differential equations [4]

(17) $$\|\underline{w}\|_0 \leqq C \, \|\operatorname{div}(\underline{v})\|_{-1}.$$

The *Grid Decomposition Property* requires that this hold on $\underline{V}_0^h$. More precisely, we have the following.

<u>Definition 1.</u> $\underline{V}_0^h$ *satisfies the GDP with constant*

(18) $$0 < C_G < \infty$$

if and only if for each

(19) $$\underline{v}_h \in \underline{V}_0^h$$

there exist

(20) $$\underline{w}_h, \underline{z}_h \in \underline{V}_0^h$$

satisfying

(21) $$\underline{v}_h = \underline{w}_h + \underline{z}_h$$

with

(22) $$\operatorname{div}(\underline{z}_h) = 0, \qquad \int_\Omega \underline{w}_h \cdot \underline{z}_h = 0, \qquad \|\underline{w}_h\|_0 \leq C_G \, \|\operatorname{div} \underline{v}_h\|_{-1} \quad .$$

Observe that GDP is related to the way div is represented in the discrete problem. Indeed, it states that if $\operatorname{div}(\underline{v}_h)$ is small for any $\underline{v}_h \in \underline{V}_0^h$, then the projection of $\underline{v}^h$ onto the orthogonal complement of

(23) $$N_h(\operatorname{div}) = \{\underline{z}^h \in \underline{V}_0^h : \operatorname{div}(\underline{z}^h)=0 \text{ in } \Omega\}$$

is also small, i.e. $\underline{z}_h$ in (21) is truly the divergence free part of $\underline{v}_h$. In the next section we shall show that GDP is *sufficient* for optimal accuracy. Here we shall show that it is *necessary* and sufficient for stability.

__Definition 2.__ *The discrete Kelvin problem is stable with constant*

$$0 < C < \infty$$

if and only if the following holds. Let

(24) $$f_h \in S_0^h$$

be given and let $\underline{v}_h$ *minimize* $\| \underline{v}^h \|_0$ *for all* $\underline{v}^h \in \underline{V}_0^h$ *satisfying*

(25) $$\operatorname{div} \underline{v}^h = f_h .$$

Then

(26) $$\| \underline{v}_h \|_0 \leqq C \| f_h \|_{-1} .$$

__Theorem 1.__ *The GDP holds with constant* C_G *if and only if the discrete Kelvin problem is stable with constant* C_G.

__Proof.__ Let $N_h(\mathrm{div})^{\perp}$ be the orthogonal complement of $N_h(\mathrm{div})$. Thus

(27) $$\underline{V}_0^h = N_h(\mathrm{div})^{\perp} \oplus N_h(\mathrm{div}) .$$

First suppose GDP holds, i.e., any $\underline{v}_h \in \underline{V}_0^h$ can be written

(28) $$\underline{v}_h = \underline{w}_h + \underline{z}_h ,$$

where

(29) $$\underline{z}_h \in N_h(\mathrm{div}), \qquad \int_\Omega \underline{w}_h \cdot \underline{z}_h = 0, \qquad \| \underline{w}_h \|_0 \leqq C_G \| \operatorname{div} \underline{v}_h \|_{-1} .$$

Moreover, let $\underline{v}_h \in \underline{V}_0^h$ satisfy

(30) $$\| \underline{v}_h \|_0 = \min. \quad \text{subject to } \underline{v}_h \in \underline{V}_0^h \text{ and } \operatorname{div}(\underline{v}_h) = f_h ,$$

where $f_h \in S_0^h = \mathrm{div}(\underset{\sim}{V}_0^h)$ is given. We want to show that

(31) $$\| \underline{v}_h \|_0 \leq C_G \| f_h \|_{-1} \quad .$$

To do this we write $\underline{v}_h$ as in (28)-(29). The claim is that $\underline{z}_h = \underline{0}$ and so

$$\| \underline{v}_h \|_0 = \| \underline{w}_h \|_0 \leq C_G \| \mathrm{div}\, \underline{v}_h \|_{-1} = C_G \| f_h \|_{-1} .$$

To see this observe that for any real number δ, $\underline{v}_h + \delta \underline{z}_h$ is in $\underset{\sim}{V}_0^h$ and

(32) $$\mathrm{div}(\underline{v}_h + \delta \underline{z}_h) = \mathrm{div}(\underline{v}_h) = f_h .$$

Thus as $\underline{v}_h$ minimizes $\| \cdot \|_0$ over $\underset{\sim}{V}_0^h$ we have

(33) $$\int_\Omega (\underline{v}_h + \delta \underline{z}_h) \cdot (\underline{v}_h + \delta \underline{z}_h) \geq \int_\Omega \underline{v}_h \cdot \underline{v}_h \quad ,$$

i.e.,

(34) $$2\delta \int_\Omega \underline{z}_h \cdot \underline{v}_h \geq -\delta^2 \int_\Omega \underline{z}_h \cdot \underline{z}_h .$$

Since δ is arbitrary we necessarily have

(35) $$\int_\Omega \underline{z}_h \cdot \underline{v}_h = 0.$$

But $\underline{v}_h = \underline{w}_h + \underline{z}_h$ and $\underline{w}_h$ is orthogonal to $\underline{z}_h$. This means

(36) $$\int_\Omega \underline{z}_h \cdot \underline{z}_h = \int_\Omega (\underline{v}_h - \underline{w}_h) \cdot \underline{z}_h = 0.$$

Conversely, assume that the Kelvin problem is stable (with constant C_G) and let $\underline{v}_h \in \underset{\sim}{V}_0^h$ be given. By (27), we can always write

(37) $$\underline{v}_h = \underline{w}_h + \underline{z}_h \quad ,$$

where

(38) $$\underline{w}_h \in N_h(\mathrm{div})^{\perp}, \quad \underline{z}_h \in \underline{N}_h(\mathrm{div}) \quad .$$

We want to select $\underline{w}_h$ such that

$$\| \underline{w}_h \|_0 \leqq C_G \| \mathrm{div}\, \underline{v}_h \|_{-1} .$$

To do this we solve a Kelvin problem. More precisely, let

$$f_h = \mathrm{div}(\underline{v}_h) \quad ,$$

and let $\underline{w}_h$ minimize $\| \underline{w}_h \|_0$ subject to

$$\underline{w}_h \in \underline{V}_0^h, \qquad \mathrm{div}(\underline{w}_h) = f_h \quad .$$

By (31) ($\underline{w}_h$ is playing the role of $\underline{v}_h$ in this inequality)

$$\| \underline{w}_h \|_0 \leqq C_G \| f_h \|_{-1}$$

and also

$$\underline{z}_h = \underline{v}_h - \underline{w}_h \in N_h(\mathrm{div}) \quad .$$

Therefore the result is proved.

In one spatial dimension (n=1) all finite element spaces satisfy GDP, the proof being exactly the same as for the space $\underline{V}_0$, i.e., (9)-(15). In two dimensions, however, this is no longer true. For example if linear elements in triangles are used, the GDP is valid for the criss-cross grid in Figure 1a but fails for the directional grids in Figure 1b and 1c. The GDP also fails for bilinear elements in the rectangles of Figure 1d. That the GDP is valid for the criss-cross grid is established in section 5.

III. Error Estimates

The major theorem of this paper is the following:

Theorem 2. *Let GDP hold with constant* c_G, *and the approximation property* [(7)-(8), *section* 2] *hold. Then there is a constant* c *depending only on* c_A *and* c_G *such that*

$$\| \underline{u}_0 - \underline{u}_h \|_0 \leq c\, h^k \| \underline{u}_0 \|_k$$

and

$$\| \phi_0 - \phi_h \|_0 \leq c\, h^{k-1} (\| \phi_0 \|_{k-1} + h \| \underline{u}_0 \|_k).$$

The key identity that will be used repeatedly is

$$(1) \qquad \int_\Omega \{ \underline{u}_0 \cdot \underline{v}^h + \phi_0 \operatorname{div}(\underline{v}^h) + \psi^h \operatorname{div}(\underline{u}_0) \} = \int_\Omega \{ \underline{u}_h \cdot \underline{v}^h + \phi_h \operatorname{div}(\underline{v}^h) + \psi^h \operatorname{div}(\underline{u}_h) \}.$$

This is valid for all $\{\psi^h, \underline{v}^h\} \in S_0^h \times \underline{V}_0^h$ (since both sides are equal to $\int_\Omega f_0 \psi^h$ by [(13), section 1] and [(6), section 2]).

Lemma 1. *For all* $\underline{w}^h \in \underline{V}_0^h$

$$(2) \qquad \| \operatorname{div}(\underline{u}_0 - \underline{u}_h) \|_0 \leq \| \operatorname{div}(\underline{u}_0 - \underline{w}^h) \|_0$$

In particular,

$$(3) \qquad \| \operatorname{div}(\underline{u}_0 - \underline{u}_h) \|_0 \leq \| \operatorname{div}(\underline{u}_0 - \underline{\hat{u}}_h) \|_0 ,$$

where $\underline{\hat{u}}_h$ *is the function in* [(7), *section* 2].

Proof. Let $\underline{v}^h = 0$ in (1). Then

$$(4) \qquad \int_\Omega \operatorname{div}(\underline{u}_0 - \underline{u}_h) \psi^h = 0$$

for all $\psi^h \in S^h = \mathrm{div}(\underline{V}^h)$. Let $\psi^h = \mathrm{div}(\underline{u}_h - \underline{w}^h)$. Then (4) gives (2).

Lemma 2.

$$(5) \qquad \| \mathrm{div}(\underline{u}_0 - \underline{u}_h) \|_{-1} \leqq C_A h \| \mathrm{div}(\underline{u}_0 - \underline{u}_h) \|_0$$

Proof. Solve

$$-\Delta\xi + \xi = \mathrm{div}(\underline{u}_0 - \underline{u}_h) \quad \text{in} \quad \Omega$$

$$\xi = 0 \quad \text{on} \quad \Gamma .$$

Then

$$(6) \qquad \| \xi \|_1 \leq \| \mathrm{div}(\underline{u}_0 - \underline{u}_h) \|_{-1} .$$

But

$$(7) \qquad \| \xi \|_1^2 = \int_\Omega \{\nabla\xi \cdot \nabla\xi + \xi^2\} = \int_\Omega \xi \, \mathrm{div}(\underline{u}_0 - \underline{u}_h).$$

We note that if $\underline{v}^h = 0$ in (1)

$$(8) \qquad \int_\Omega \psi^h \mathrm{div}(\underline{u}_0 - \underline{u}_h) = 0 \quad \text{for all} \quad \psi^h \in S_0^h .$$

Thus letting $\psi^h = \hat{\xi}^h$

$$(9) \qquad \| \xi \|_1^2 = \int_\Omega (\xi - \hat{\xi}^h) \mathrm{div}(\underline{u}_0 - \underline{u}_h)$$

$$\leqq \| \xi - \hat{\xi}^h \|_0 \quad \| \mathrm{div}(\underline{u}_0 - \underline{u}_h) \|_0 .$$

Using the approximation property [(8), section 2] with $k = 1$ we can choose $\hat{\xi}^h$ such that

(10) $$\| \xi - \hat{\xi}^h \|_0 \leq C_A h \| \xi \|_1$$

Thus (5) follows from (6), (9), and (10).

Observe that $\operatorname{div}(\underline{u}_0 - \underline{u}_h)$ is optimal in $\| \cdot \|_{-1}$, i.e.

(11) $$\| \operatorname{div}(\underline{u}_0 - \underline{u}_h) \|_{-1} \leq C_A h \| \operatorname{div}(\underline{u}_0 - \underline{u}_h) \|_0 \qquad \text{(Lemma 2)}$$

$$\leq C_A h \| \operatorname{div}(\underline{u}_0 - \underline{\hat{u}}_h) \|_0 \qquad \text{(Lemma 1)}$$

$$\leq C_A h \| \underline{u}_0 - \underline{\hat{u}}_h \|_1$$

$$\leq C_A^2 h^k \| \underline{u}_0 \|_k . \qquad \text{([7], Section 2)}$$

So far GDP has not been used, however from this point on it will play a crucial role. In particular, write

(12) $$\underline{u}_h - \underline{\hat{u}}_h = \underline{w}_h + \underline{z}_h$$

where

(13) $$\operatorname{div}(\underline{z}_h) = 0, \quad \int_\Omega \underline{w}_h \cdot \underline{z}_h = 0, \quad \| \underline{w}_h \|_0 \leq C_G \| \operatorname{div}(\underline{u}_h - \underline{\hat{u}}_h) \|_{-1} .$$

Note that for all $\underline{v} \in \underline{V}$

$$\| \operatorname{div} \underline{v} \|_{-1} \leq \| \underline{v} \|_0 .$$

Indeed

$$\| \operatorname{div} \underline{v} \|_{-1} = \sup_{\psi \in H_0^1(\Omega)} \frac{\int_\Omega (\operatorname{div} \underline{v}) \psi}{\| \psi \|_1} = \sup_{\psi \in H_0^1(\Omega)} \frac{-\int_\Omega \underline{v} \cdot \nabla \psi}{\| \psi \|_1}$$

$$\leq \sup_{\psi \in H_0^1(\Omega)} \frac{\| \underline{v} \|_0 \, \| \nabla \psi \|_0}{\| \psi \|_1} \leq \| \underline{v} \|_0 .$$

Thus

$$\| \underline{w}_h \|_0 \leq C_G \{ \| \operatorname{div}(\underline{u}_0 - \underline{\hat{u}}_h) \|_{-1} + \| \operatorname{div}(\underline{u}_0 - \underline{u}_h) \|_{-1} \}$$

and so

$$(14) \qquad \| \underline{w}_h \|_0 \leq C_G \{ \| \underline{u}_0 - \hat{\underline{u}}_h \|_0 + \| \mathrm{div}(\underline{u}_0 - \underline{u}_h) \|_{-1} \}.$$

Thus it is sufficient to obtain a similar bound for $\underline{z}_h$. In particular, letting $\psi^h = 0$ and $\underline{v}^h = \underline{z}_h$ in (1) we obtain

$$(15) \qquad \int_\Omega \underline{u}_h \cdot \underline{z}_h = \int_\Omega \underline{u}_0 \cdot \underline{z}_h$$

and so

$$(16) \qquad \int_\Omega \underline{z}_h \cdot \underline{z}_h = \int_\Omega (\underline{u}_h - \hat{\underline{u}}_h) \cdot \underline{z}_h = \int_\Omega (\underline{u}_0 - \hat{\underline{u}}_h) \cdot \underline{z}_h \; .$$

This gives

$$(17) \qquad \| \underline{z}_h \|_0 \leq \| \underline{u}_0 - \hat{\underline{u}}_h \|_0 \; ;$$

then (11), (12), (14) and (17) give

$$(18) \qquad \| \underline{u}_h - \hat{\underline{u}}_h \|_0 \leq C h^k \| \underline{u}_0 \|_k$$

and from the triangle inequality we obtain the first part of Theorem 2,

$$\| \underline{u}_h - \underline{u}_0 \|_0 \leq C h^k \| \underline{u}_0 \|_k \; .$$

To estimate $\phi_0 - \phi_h$ we let $\psi^h = 0$ in (1) to get

$$(19) \qquad \int_\Omega \{ \phi_h \mathrm{div}\, \underline{v}^h \} = \int_\Omega \{ \phi_0 \, \mathrm{div}\, \underline{v}^h + \underline{v}^h \cdot (\underline{u}_0 - \underline{u}_h) \} \; .$$

Let $\hat{\phi}_h \in S_0^h$. Then

$$(20) \qquad \int_\Omega \{ (\phi_h - \hat{\phi}_h) \mathrm{div}\, \underline{v}^h \} = \int_\Omega \{ (\phi_0 - \hat{\phi}_h) \mathrm{div}(\underline{v}^h) + \underline{v}^h \cdot (\underline{u}_0 - \underline{u}_h) \}.$$

Now let $\hat{\phi}_h$ be the function in [(8), section 2] with $\phi = \phi_0$. Since $S_0^h = \mathrm{div}(\underline{V}_0^h)$ there is a $\underline{v}_h \in \underline{V}_0^h$ such that

$$(21) \qquad \phi_h - \hat{\phi}_h = \mathrm{div}(\underline{v}_h) .$$

We now use GDP to write

$$\underline{v}_h = \underline{w}_h + \underline{z}_h \quad , \tag{22}$$

with

$$\operatorname{div}(\underline{z}_h) = 0, \quad \int_\Omega \underline{w}_h \cdot \underline{z}_h = 0, \quad \| \underline{w}_h \|_0 \leq C_G \| \phi_h - \hat{\phi}_h \|_{-1} \, . \tag{23}$$

Letting $\underline{v}^h = \underline{w}_h$ in (20) we obtain

$$\begin{aligned} \int_\Omega |\phi_h - \hat{\phi}_h|^2 &\leq \| \phi_0 - \hat{\phi}_h \|_0 \| \phi_h - \hat{\phi}_h \|_0 + \| \underline{w}_h \|_0 \| \underline{u}_0 - \underline{u}_h \|_0 \\ &\leq \| \phi_0 - \hat{\phi}_h \|_0 \| \phi_h - \hat{\phi}_h \|_0 + C_G \| \phi_h - \hat{\phi}_h \|_0 \| \underline{u}_0 - \underline{u}_h \|_0 \, . \end{aligned} \tag{24}$$

Thus

$$\| \phi_h - \hat{\phi}_h \|_0 \leq \| \phi_0 - \hat{\phi}_h \|_0 + C_G \| \underline{u}_0 - \underline{u}_h \|_0 \tag{25}$$

The second part of Theorem 2 now follows from an application of the triangle inequality. Thus Theorem 2 is proved.

With linear elements on the criss-cross grid, Theorem 2 asserts that the L_2 error in $(\underline{u}_0 - \underline{u}_h)$ is of $0(h^2)$. This sharpens the $0(h)$ estimate found in [2] and [5]. The L_2 error in $(\phi_0 - \phi_h)$ is $0(h)$, the same as predicted in [2] and [5]. However, if in (20) we choose $\hat{\phi}_h$ to be the best L_2 approximations in S_0^h to ϕ_0, then, since $\operatorname{div}(\underline{v}^h) \in S_0^h$, the first term on the right hand side of (20) vanishes. We are then led to the conclusion that

$$\| \phi_h - \hat{\phi}_h \|_0 = 0(h^2), \tag{26}$$

i.e. the mean value of ϕ_0 over a given triangle is actually approximated to $0(h^2)$. This phenomena is illustrated in the numerical examples of section 4.

IV. Numerical Results

In this section we briefly report the results of computations based on the Kelvin principle. These results give evidence of the essential role played by the GDP. The examples of this section deal with the Poisson equation [(1), section 1]. An equivalent first order system is given by [(4) - (5), section 1].

We first consider results for the mixed data problem depicted in Figure 2a using the directional grid illustrated in Figure 1b. The particular problem considered has an exact solution given by

$$(1) \qquad \phi = \sin(\pi x/2)\cos(\pi y) \ .$$

Figure 3 displays the L_2 error of the approximate solution for ϕ and the components u and v of $\underline{u} = \text{grad}\,\phi$. The figure indicates that the L_2 errors in u and v remains roughly constant and the L_2 error in ϕ grows linearly as the size of the grid is reduced. We recall from section 2 that the GDP is necessary and sufficient for the stability of the Kelvin approximation. The results shown in Figure 3 indicate that for the directional grid the "constant" C_G appearing in the definition of the GDP in fact grows like h^{-2}, where h is a measure of the grid size. As a result, all accuracy in the approximation to $\underline{u}$ is lost, and the approximation in ϕ actually becomes unbounded. These results, and those below concerning the criss-cross grid give evidence of the importance of the GDP.

The directional grid used to generate the results of Figure 3 does not satisfy the GDP. However, Lemma 1 of section 3 is independent of this property of the grid. In the context of the directional grid, that lemma shows that the divergence of the error in the approximation to $\underline{u}$ should be $O(h)$. This result is confirmed in Figure 3 where that divergence is graphed as a function of h. As is evident from the figure, the divergence of the error in $\underline{u}$ is indeed $O(h)$ even though the error in $\underline{u}$ itself is $O(1)$.

We now consider results using the "criss-cross" grid illustrated in Figure 1a. Figure 4 displays the L_2 errors of the approximate solutions for u and v. Results are given for the mixed data problem with exact solution given by (1) and for a Dirichlet problem (see Figure 2b) with exact solution

$$\phi = \sin(\pi x)\sin(\pi y) .$$

The mixed data and Dirichlet problems were approximated using an evenly spaced grid. In addition, computations for the mixed data problem were carried out using a variable grid whose spacing is determined by choosing an even spacing in a (ξ,η) coordinate system, and then letting

$$x = \xi^3 \qquad \text{and} \qquad y = \eta^3 .$$

This stretching has the effect of accumulating grid points near $x = 0$ and $y = 0$. For all cases, the computed rate of convergence, using criss-cross grids, is of second order. The results shown in Figure 4, especially when compared with those of Figure 3 for the directional grid, are lucid evidence of the necessity of the GDP to the achievement of optimal orders of accuracy.

Also shown in Figure 4 are the values of $\| \phi_h - \hat{\phi}_h \|_0$ for the problems described above, confirming the result (26) of section 3.

V. Proof that the Criss-Cross Grid Satisfies the GDP

For simplicity consider the Dirichlet problem for the uniform grid shown in Figure 1a with $\underline{V}_0^h = \underline{V}^h$ being the space of $\mathbb{R}^2$ - valued piecewise linear functions. No assumptions on Ω will be required. To verify that this grid satisfies the GDP we must show that there is a positive number

$$0 < C_G < \infty \tag{1}$$

independent of h for which the following holds. Given any

$$f_h \in S^h = \operatorname{div}(\underline{V}^h) \tag{2}$$

there is a $\underline{v}_h$ in $\underline{V}^h$ for which

$$\operatorname{div}(\underline{v}_h) = f_h \tag{3}$$

and

$$\| \underline{v}_h \|_0 \leqq C_G \| f_h \|_{-1} . \tag{4}$$

Since $\underline{V}^h$ consists of piecewise linear functions on the grid in Figure 1a, observe that (2) implies that each f_h in S^h is a piecewise constant function. What is interesting is that S^h is a *strict* subspace of the space $\hat{S}^h$ of all piecewise constant functions on the criss-cross grid in Figure 1a. Indeed, the following gives a rule for determining when a function f_h in $\hat{S}^h$ is actually in S^h.

Lemma 3: *Let* f *be in* $\hat{S}^h$. *Then* f *is in* S^h *if and only if for any rectangle* R *(see Figure 5)*

$$f_1 + f_3 = f_2 + f_4 \quad , \tag{5}$$

where f_j *is the value of* f *in* T_j.

Proof. We must construct continuous piecewise linear functions u,v such that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = f \quad , \tag{6}$$

in each triangle. To do this we close the system with

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = g \quad , \tag{7}$$

where the piecewise constant function g is to be determined.

Observe that (6) - (7) is hyperbolic, and we shall solve it by the method of characteristics. The characteristic coordinates are

$$\xi = x - y \quad , \qquad \eta = x + y \quad , \tag{8}$$

and letting

$$2U = (u - v) \quad , \qquad 2V = (u + v) \quad , \tag{9}$$

we obtain

$$\text{(10)} \qquad \frac{\partial U}{\partial \xi} = f - g \quad , \quad \frac{\partial V}{\partial \eta} = f + g \quad .$$

Let the arbitrary rectangle R in Figure 5 be given. We first construct U,V,g in R. Following this we show that they can be globally extended such that $\{u,v\}$ defined by (9) is in $\underline{V}^h$, i.e., it is continuous in Ω as well as linear in each triangle.

Since f and g are constants in each triangle T_j $(j = 1,\ldots,4)$, then any function U satisfing the first equation in (10) will be continuous in R if and only if

$$\text{(11)} \qquad f_1 - g_1 = f_4 - g_4 \quad , \quad f_3 - g_3 = f_2 - g_2 \quad ,$$

where f_j, g_j are the values of f,g in the triangle T_j. Similarly, continuity of V requires

$$\text{(12)} \qquad f_3 + g_3 = f_4 + g_4 \quad , \quad f_1 + g_1 = f_2 + g_2 \quad .$$

It follows immediately from (11) - (12) that (5) is a necessary condition for (11) - (12) to have a solution g_i; moreover, it is also sufficient. Indeed, let

$$\text{(13)} \qquad g_4 = \text{arbitrary} \quad ,$$

then

$$\text{(14)} \qquad g_1 - g_4 = f_1 - f_4, \quad g_2 - g_4 = f_1 - f_3, \quad g_3 - g_4 = f_4 - f_3 \quad ,$$

is a solution provided (5) holds.

To define U, V globally we first of all define the piecewise constant function g in each rectangle so that (11) - (12) holds. To construct U and V we simply integrate (10) along the characteristics working from rectangle to rectangle. In particular, consider Figure 6 where for simplicity the region Ω is shown as a rectangle. Along the left and top sides U can be taken as an arbitrary linear function. To determine its values in a given rectangle R we simply integrate (10) from points A to B as shown in Figure 6. The conditions (11) - (12) insure U is a continuous function in R, and using the value of U so obtained at B to start the integration in the next box, interelement continuity is assured.

Since f and g are constants in each triangle, U and V will be linear functions of ξ, η in each triangle. Hence u and v defined by (9) will be continuous piecewise linear functions (i.e., $\{u,v\}$ in $\underline{V}^h$).

Note that since the dimension $\dim(\hat{S}^h)$ of $\hat{S}^h$ is equal to the number m of triangles in the grid, it follows from (5) that

$$\dim S^h = 3m/4.$$

Moreover, a locally defined basis can be constructed as follows. For each rectangle R (see Figure 5) we associate three functions ψ_1, ψ_2, ψ_3 which vanish outside R. The piecewise constant function ψ_i is uniquely determined in R by the requirement that it is identically 1 in $T_i \cup T_{i+1}$ and zero in the other two triangles in R. As R varies over all rectangles this process defines $3m/4$ independent functions in S^h and hence the set of such functions is a basis for S^h. Interestingly, this shows that S^h is the linear hull of the union of the piecewise constant spaces associated with the directional grids shown in Figures 1b and 1c. Therefore the approximation property [(8), section 2] is certainly valid for the above choice of S^h.

We now return to the proof of (4), which is contained in the following result.

Lemma 4. *There is a number* $0 < c_G < \infty$ *independent of* h *such that for each* f_h *in* S^h *we have*

$$f_h = \mathrm{div}[\underline{v}_h], \tag{15}$$

where $\underline{v}_h$ *in* $\underline{V}^h$ *satisfies*

$$\| \underline{v}_h \|_0 \leqq c_G \| f_h \|_{-1} . \tag{16}$$

Proof. To simplify notation we drop all subscripts involving h since all functions that will be encountered will be in S^h or V^h. As in Lemma 3 we work in the rotated coordinates (ξ,η) defined by (8). In addition we order the vertices in a sequential manner starting at the bottom of the region and moving left to right as in Figure 7. Observe that the center of each (rotated) rectangle has an index (α,β), where $\alpha+\beta$ is an integer, while $\gamma+\delta+\frac{1}{2}$ is integral for the indices (γ,δ) of the corner points. We denote the rectangle whose centroid has index (α,β) by $R^{\alpha,\beta}$ and let $T_k^{\alpha,\beta}(k=1,2,3,4)$ denote the four enclosed triangles.

Given f in S^h we must construct continuous piecewise linear functions u and v such that

$$f = \mathrm{div}\binom{u}{v} = \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} . \tag{17}$$

We let $u^{\alpha,\beta}$, $v^{\alpha,\beta}$ denote the values of u,v at the vertices, and let $f_k^{\alpha,\beta}$ denote the value of f in the triangle $T_k^{\alpha,\beta}$. Then a direct calculation gives

$$f_2^{\alpha,\beta} = D_1^+ u + D_2^+ v, \tag{18}$$

where

$$(19) \qquad D_1^+u = \frac{u^{\alpha+\frac{1}{2},\beta} - u^{\alpha,\beta}}{h}, \qquad D_2^+v = \frac{v^{\alpha,\beta+\frac{1}{2}} - v^{\alpha,\beta}}{h}.$$

Similarly,

$$(20) \qquad \begin{aligned} f_1^{\alpha,\beta} &= D_1^-u + D_2^+v \\ f_3^{\alpha,\beta} &= D_1^+u + D_2^-v \\ f_4^{\alpha,\beta} &= D_1^-u + D_2^-v, \end{aligned}$$

where the difference operators D_1^-, D_2^- are defined by

$$(21) \qquad D_1^-u = \frac{u^{\alpha,\beta} - u^{\alpha-\frac{1}{2},\beta}}{h}, \qquad D_2^-v = \frac{v^{\alpha,\beta} - v^{\alpha,\beta-\frac{1}{2}}}{h}.$$

Observe that (18) and (20) can be combined into

$$(22) \qquad \frac{u^{\alpha+\frac{1}{2},\beta} - u^{\alpha-\frac{1}{2},\beta}}{2h} + \frac{v^{\alpha,\beta+\frac{1}{2}} - v^{\alpha,\beta-\frac{1}{2}}}{2h} = \tfrac{1}{2}(f_1^{\alpha,\beta} + f_3^{\alpha,\beta}) = \tfrac{1}{2}(f_2^{\alpha,\beta} + f_4^{\alpha,\beta}),$$

a relation which reconfirms the necessity of the condition (5). We rewrite (22) as

$$(23) \qquad \operatorname{div}_h \binom{u}{v} = \overline{f}^{\alpha,\beta},$$

where div_h denotes the difference operator on the left hand side and $\overline{f}^{\alpha,\beta}$ denotes the average of f on the right.

Observe that (22) (or (23)) involves values of u and v only at the corner points of the rectangles (i.e., vertices (γ,δ) where $\gamma + \delta + \frac{1}{2}$ is integral). Once these have been determined the values at the centroids of the rectangles (i.e., vertices (α,β) where $\alpha+\beta$ is integral) are then given by

$$(24)\qquad u^{\alpha,\beta} = \tfrac{1}{2}(u^{\alpha+\frac{1}{2},\beta} + u^{\alpha-\frac{1}{2},\beta}) + \frac{h}{2}(f_1^{\alpha,\beta} - f_2^{\alpha,\beta}),$$

$$(25)\qquad v^{\alpha,\beta} = \tfrac{1}{2}(v^{\alpha,\beta+\frac{1}{2}} + v^{\alpha,\beta-\frac{1}{2}}) + \frac{h}{2}(f_3^{\alpha,\beta} - f_2^{\alpha,\beta}).$$

That is, (23) - (25) are three independent relations among the four dependent equations (18), (20).

To solve (23) we introduce a discrete potential θ satisfying

$$(26)\qquad u^{\alpha+\frac{1}{2},\beta} = \frac{\theta^{\alpha+1,\beta} - \theta^{\alpha,\beta}}{2h}, \qquad v^{\alpha,\beta+\frac{1}{2}} = \frac{\theta^{\alpha,\beta+1} - \theta^{\alpha,\beta}}{2h}.$$

Then (23) is equivalent to

$$(27)\qquad \left(\frac{1}{4h^2}\right)\left(\left[\theta^{\alpha+1,\beta} - 2\theta^{\alpha,\beta} + \theta^{\alpha-1,\beta}\right] + \left[\theta^{\alpha,\beta+1} - 2\theta^{\alpha,\beta} + \theta^{\alpha,\beta-1}\right]\right) = \overline{f}^{\alpha,\beta}.$$

Observe that this equation has a "red-black" decoupling. Indeed, only values of θ at the centroids of rectangles (i.e., vertices (α,β) where $\alpha+\beta$ is integral) are involved. Moreover, there are two types of such points. The first are "red" rectangles $R^{\alpha,\beta}$ where α and β are both integers $(\alpha = i,\ \beta = j)$. The second are "black" rectangles where $\alpha = i+\frac{1}{2}$, $\beta = j+\frac{1}{2}$.

Since all boundary conditions are natural we can extend the grid to cover Ω and let $\theta^{\alpha,\beta} = 0$ outside Ω. Then (27) becomes a standard five point star on the red rectangular grid, and a standard five point star on the black rectangular grid. Moreover, defining u,v by (26) we get the standard estimate

$$(28) \qquad h^2 \sum_{\alpha,\beta} \left\{ |u^{\alpha+\frac{1}{2},\beta}|^2 + |v^{\alpha,\beta+\frac{1}{2}}|^2 \right\} \leqq C \, \| f \|_{-1}^2 ,$$

for some absolute constant $0 < C < \infty$. In addition, defining u,v at the centroids of rectangles by (24) - (25), we get

$$(29) \qquad h^2 \sum_{\alpha,\beta} \left\{ |u^{\alpha,\beta}|^2 + |v^{\alpha,\beta}|^2 \right\} \leqq C(\| f \|_{-1}^2 + h^2 \, \| f \|_0^2),$$

where the sum is over all vertices (α,β). Letting u,v be the continuous piecewise linear functions whose values at the vertex (α,β) is $u^{\alpha,\beta}$, $v^{\alpha,\beta}$, we get

$$(30) \qquad \| u \|_0^2 + \| v \|_0^2 \leqq C(\| f \|_{-1}^2 + h^2 \, \| f \|_0^2).$$

Finally for the uniform grid being considered we have the inverse inequality for function $f \in S^h$ [3]

$$(31) \qquad \| f \|_0 \leqq (C/h) \| f \|_{-1} ;$$

hence (15) - (16) hold with

$$\underline{v}_h = \binom{u}{v}.$$

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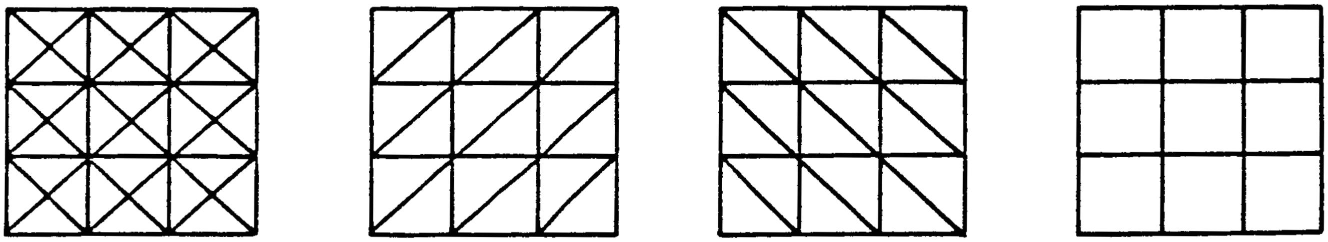

(a) (b) (c) (d)

Figure 1: Grids.
a) Criss-cross triangles
b-c) Directional triangles
d) Bilinear quadrilaterals

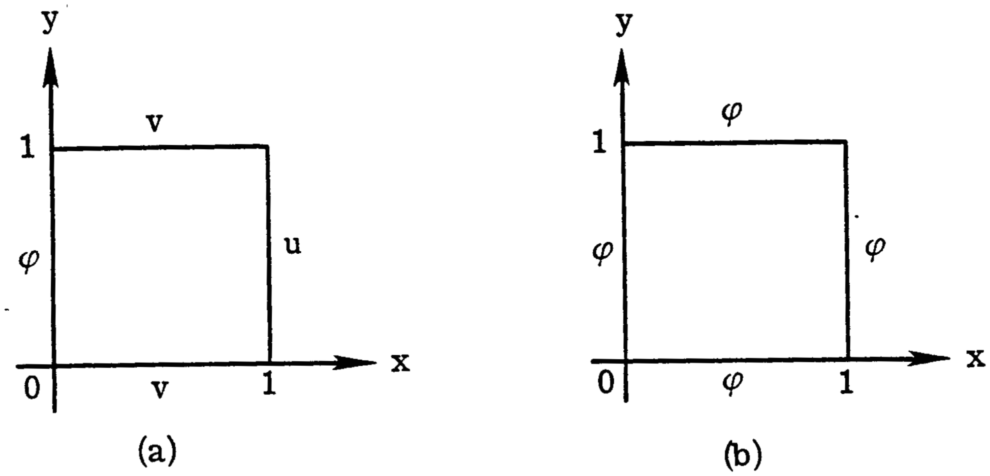

Figure 2.- Boundary value specifications used in numerical examples. (a) Mixed data; (b) Dirichlet data.

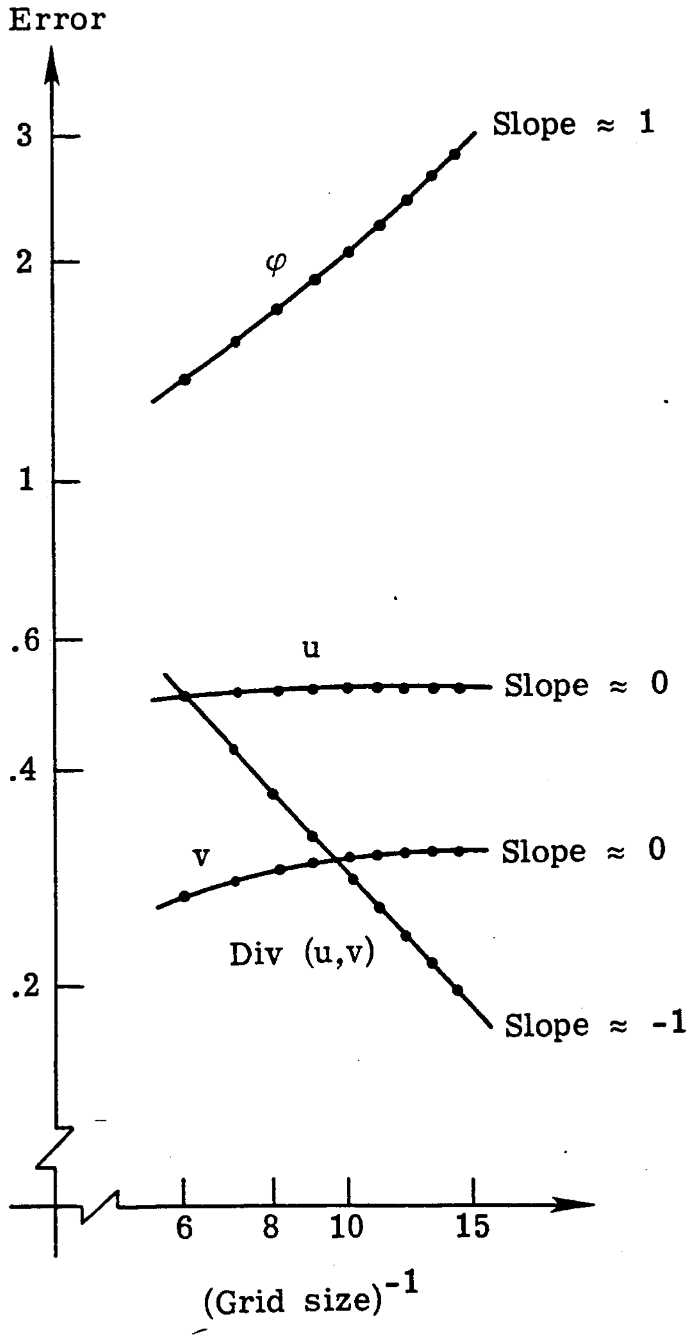

Figure 3.- L_2 error in the Kelvin approximation to ϕ, $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$, and $\mathrm{div}(u,v) = \partial u/\partial x + \partial v/\partial y$ using the directional grid for the mixed data problem.

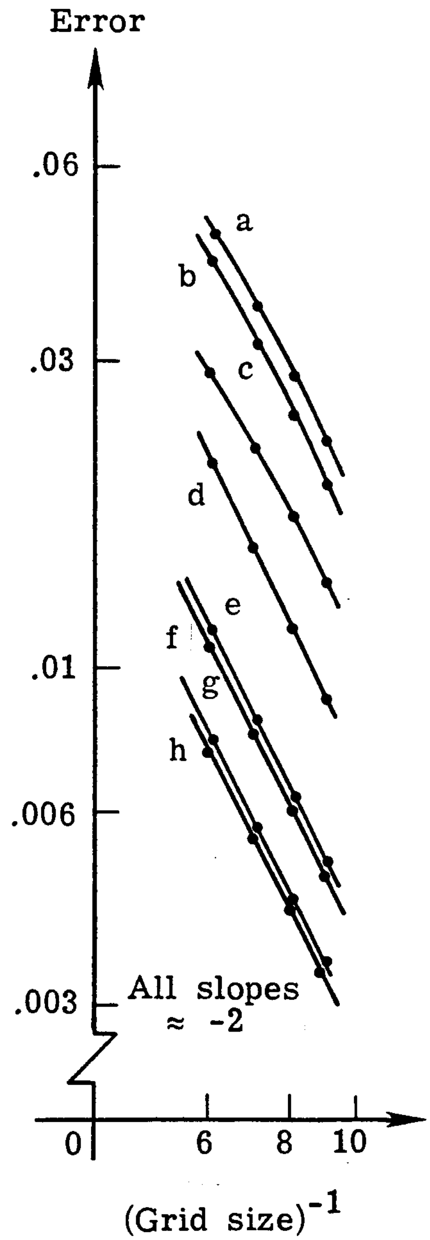

Figure 4.- L_2 error in the Kelvin approximation to $u = \partial\phi/\partial x$ and $v = \partial\phi/\partial y$ and the L_2 norm of the difference in the Kelvin approximation to ϕ and the best L_2 approximation to ϕ, using criss-cross grid. (a,d,f) displays u; (b,d,h) displays v; (c,e,g) displays ϕ. (a,b,c) for the mixed data problem using a variable grid; (d,e) for the Dirichlet data problem using a regular grid; (f,g,h) for the mixed data problem on a regular grid.

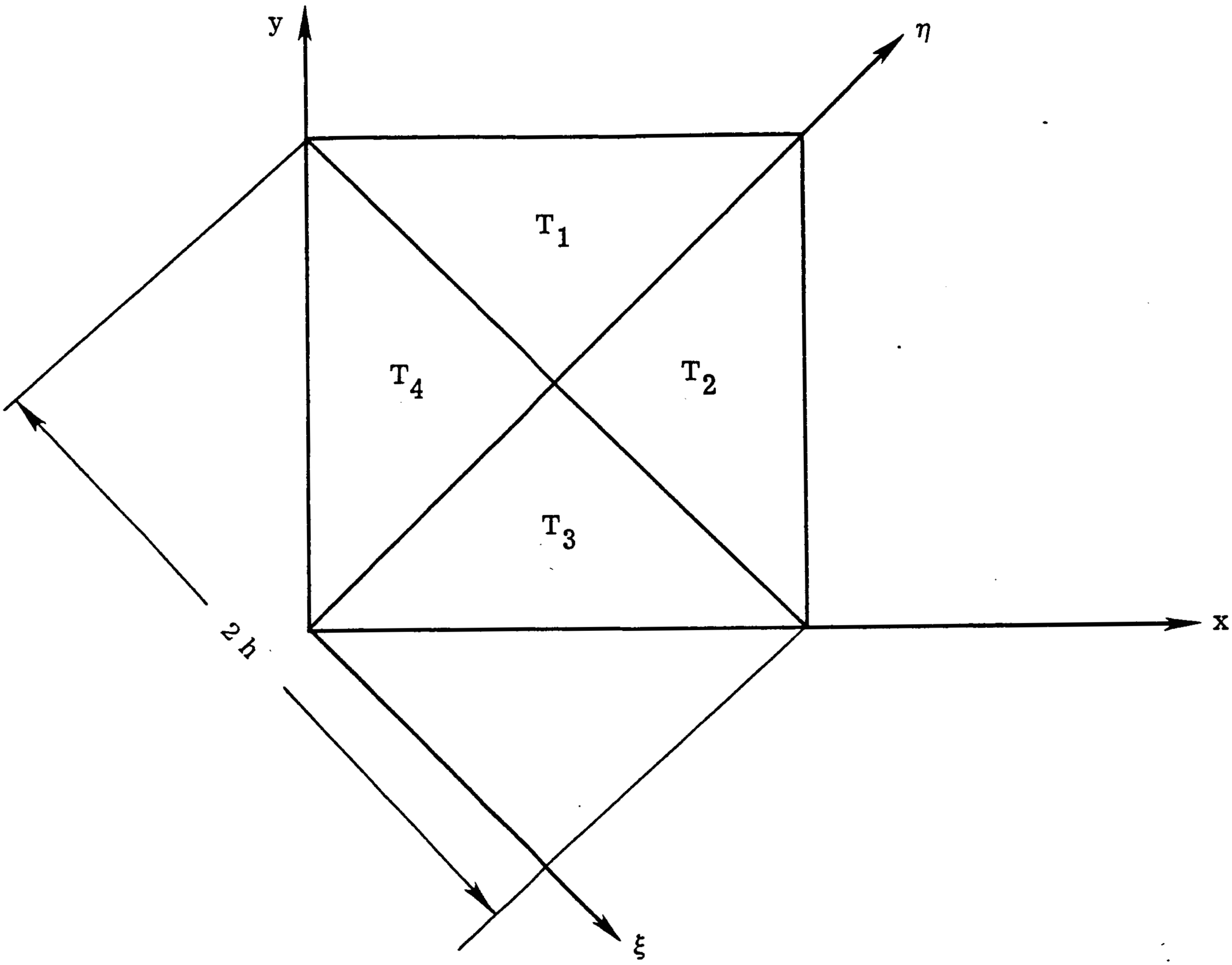

Figure 5.- Generic rectangle R.

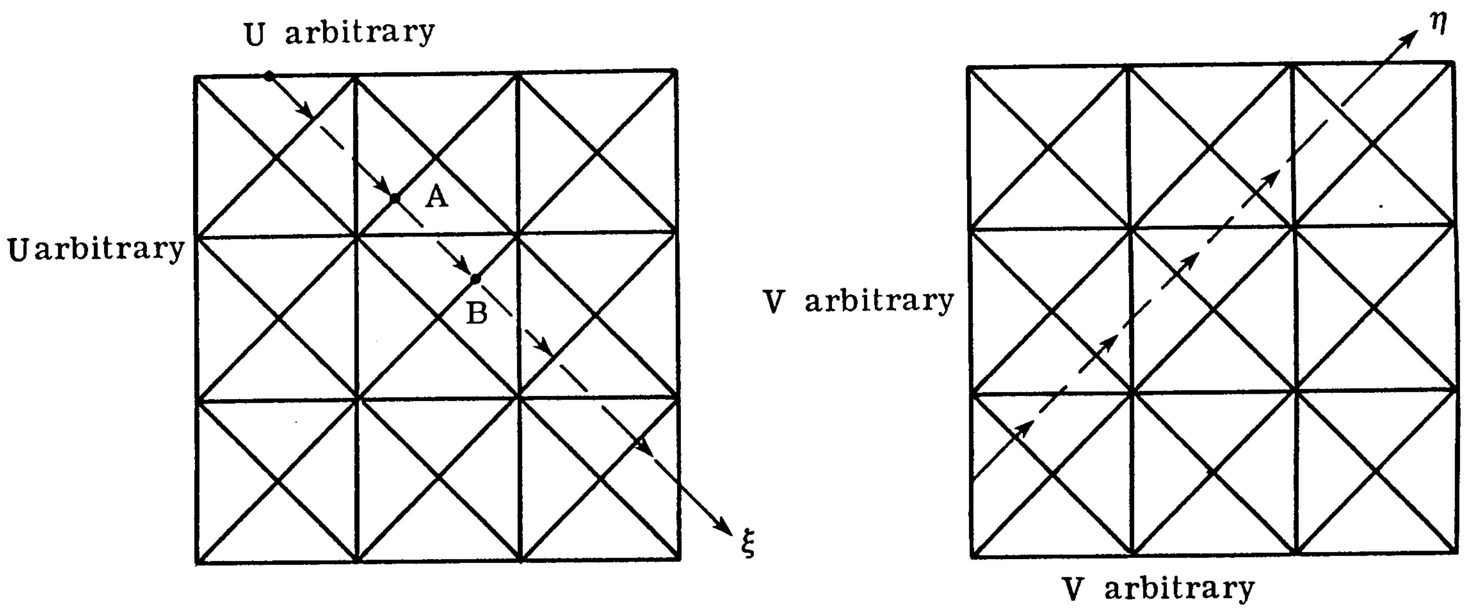

Figure 6.- Characteristic directions for U and V.

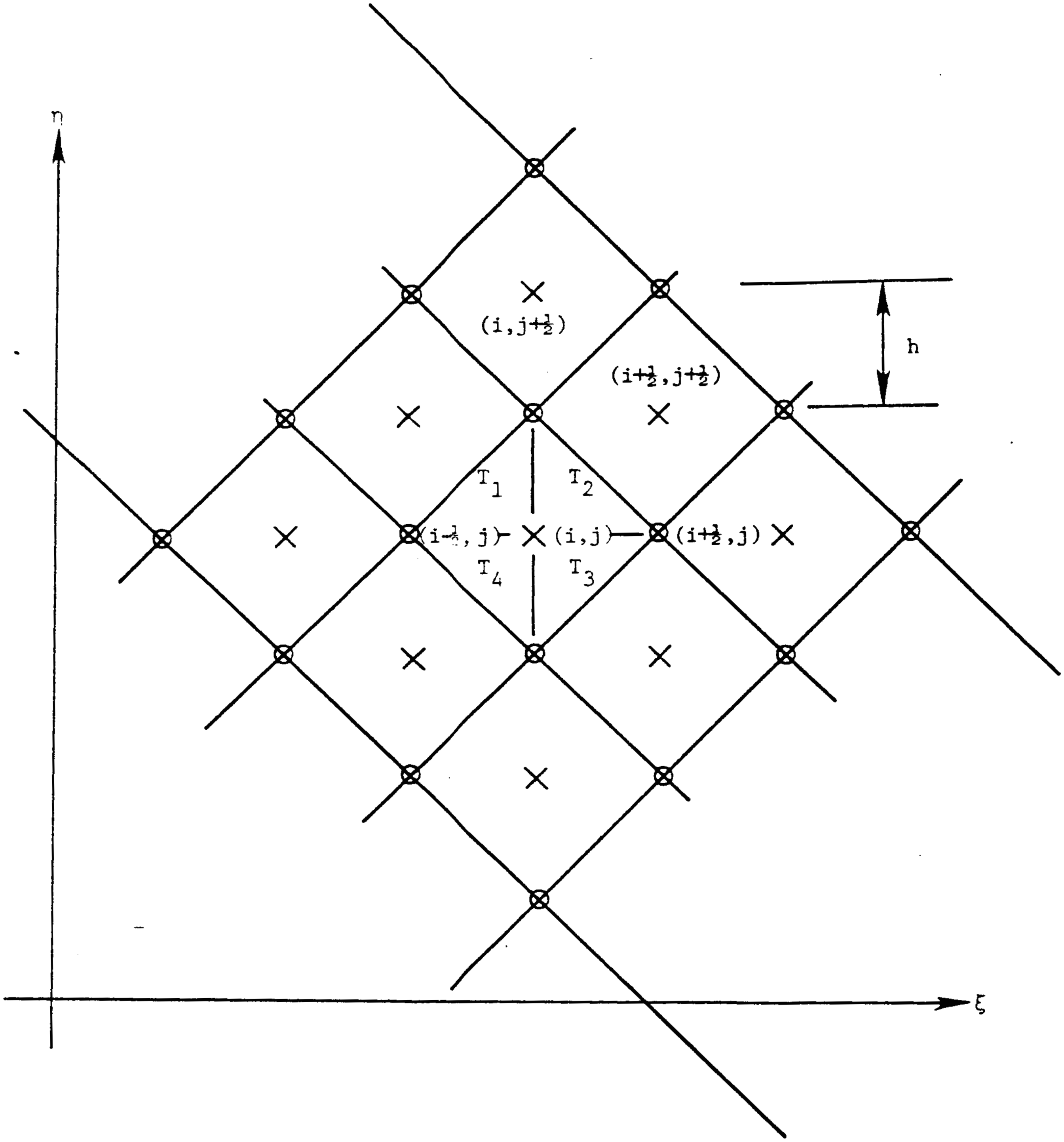

Figure 7. Ordering of vertices and triangles for Lemma 4.